ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Algebra 256 (2002) 568–582

JOURNAL OF Algebra

www.academicpress.com

Substructures of bi-Frobenius algebras

Yukio Doi *

Department of Mathematics, Fukui University, Fukui 910, Japan

Received 10 October 2001

Communicated by Susan Montgomery

Abstract

Bi-Frobenius algebras (or bF algebras) were recently introduced by the author and Takeuchi. These are both Frobenius algebras and Frobenius coalgebras and satisfy some compatibility conditions. The concept generalizes finite dimensional Hopf algebras. In Section 1 we give conditions for finite dimensional algebras and coalgebras to be bF algebras. In Section 2 we discuss substructures, quotient structures of bF algebras. Section 3 is devoted a study of morphisms and we deduce some results in Koppinen's theory.

1. Introduction

A bi-Frobenius algebra (or bF algebra) is a finite dimensional algebra and coalgebra H together with $\phi \in H^*$, $t \in H$, and the induced endomorphism $S: H \to H$, $h \mapsto \sum \phi(t_1 h) t_2$ satisfying some specific conditions (see Definition 1.1). In Section 1 we give several conditions for finite dimensional algebras and coalgebras to be bF algebras (Lemma 1.2 and Theorem 1.8).

Let H be a Hopf algebra with antipode σ. If $K \subset H$ is a subalgebra and subcoalgebra with $\sigma(K) \subset K$, then K is itself a Hopf algebra with the induced structures. However, this is not true for bF algebras in general (see Example 2.3). For any bF algebra $H = (H, \phi, t, S)$ we consider when $K \subset H$ is a bF algebra. We give a useful criterion (Theorem 2.1) whenever K has a convolution inverse of id, σ (called a Hopf antipode).

* Current address: Okayama University, Faculty of Education, Okayama 700-8530, Japan.

0021-8693/02/$ – see front matter
PII: S0021-8693(02)00143-6

We call $\phi \otimes t$ ($\in H^* \otimes H$) the integral double for H. The integral double for a bF subalgebra K is described from one for H by using σ and a linear projection $\pi : H \to K$ (see formula (7)). In Theorem 2.5 we give a description of the integral double in terms of the trace function on $\mathrm{End}(H)$.

We also consider in Theorem 2.4 when a quotient H/I has a bF algebra structure, using old Nakayama's theorem. In final section we consider certain types of morphisms for bF algebras. We deduce several variants of Koppinen's results [Ko2,Ko3].

Throughout k is a field and all vector spaces are over k. Let H be an algebra and coalgebra over k. For $h \in H$ we represent $\Delta(h) \in H \otimes H$ by $\Delta(h) = \sum h_1 \otimes h_2$. Observe that the dual algebra $H^* = \mathrm{Hom}(H, k)$ has a two-sided H-module structure

$$(h \rightharpoonup f)(x) = f(xh) \quad \text{and} \quad (f \leftharpoonup h)(x) = f(hx)$$

for all $h, x \in H$ and $f \in H^*$, and H has a two-sided H^*-module structure

$$f \rightharpoonup h = \sum h_1 f(h_2) \quad \text{and} \quad h \leftharpoonup f = \sum f(h_1) h_2$$

for all $f \in H^*$ and $h \in H$.

We begin to recall the definition of bF algebras [DT].

Definition 1.1. Let H be a finite dimensional algebra and coalgebra over a field k, $\phi \in H^*$, $t \in H$. Define a map S by

$$S : H \to H, \quad S(h) = t \leftharpoonup (h \rightharpoonup \phi) = \sum \phi(t_1 h) t_2. \tag{1}$$

(H, ϕ, t, S) is called a *bi-Frobenius algebra* (*or bF algebra*) if

(BF1) the counit $\varepsilon_H \in \mathrm{Alg}(H, k)$;
(BF2) $1_H \in G(H)$;
(BF3) (H, ϕ) is a Frobenius algebra, i.e., $\phi \leftharpoonup H = H^*$;
(BF4) (H, t) is a Frobenius coalgebra, i.e., $t \leftharpoonup H^* = H$;
(BF5) S is an anti-algebra map, i.e., $S(hh') = S(h')S(h)$, $S(1) = 1$;
(BF6) S is an anti-coalgebra map, i.e., $\Delta(S(h)) = \sum S(h_2) \otimes S(h_1)$, $\varepsilon(S(h)) = \varepsilon(h)$.

S is called the *bF antipode*, due to these properties. It does not mean a convolution inverse of identity. From (BF3) and (BF4) we have four linear isomorphisms:

$$\theta : H \simeq H^*, \quad h \mapsto \phi \leftharpoonup h, \qquad \kappa : H^* \simeq H, \quad f \mapsto t \leftharpoonup f,$$
$$\theta' : H \simeq H^*, \quad h \mapsto h \rightharpoonup \phi, \qquad \kappa' : H^* \simeq H, \quad f \mapsto f \rightharpoonup t.$$

We have from [DT, 2.1],

$$S = \kappa \circ \theta', \qquad \mathcal{N} = \theta'^{-1} \circ \theta, \qquad {}^{c}\mathcal{N} = \kappa' \circ \kappa^{-1} \tag{2}$$

where $\mathcal{N}$ is the Nakayama automorphism of (H,ϕ), i.e., $\phi(xy)=\phi(y\mathcal{N}(x))$ and ${}^c\mathcal{N}$ is the co-Nakayama automorphism of (H,t), i.e., $\sum {}^c\mathcal{N}(t_2)\otimes t_1=\sum t_1\otimes t_2$.

In particular, the bF antipode S is a bijection. We denote by $\overline{S}$ the composite inverse of S.

Lemma 1.2. *Let H be a finite dimensional algebra and coalgebra satisfying* (BF1) *and* (BF2). *Assume that there exist $\phi\in H^*$ and $t\in H$ such that the map*

$$S: H\to H,\quad h\mapsto \sum\phi(t_1h)t_2$$

is an anti-algebra and anti-coalgebra automorphism. Then (H,ϕ,t,S) becomes a bF algebra.

Proof. By assumption we have for any $f\in H^*$,

$$\phi \leftharpoonup \Bigl(\sum f\bigl(\overline{S}(t_2)\bigr)t_1\Bigr)=f,$$

since

$$\begin{aligned}\phi\Bigl(\sum f\bigl(\overline{S}(t_2)\bigr)t_1h\Bigr) &= \sum f\bigl(\overline{S}(t_2)\bigr)\phi(t_1h)=f\Bigl(\overline{S}\Bigl(\sum\phi(t_1h)t_2\Bigr)\Bigr)\\ &= f\bigl(\overline{S}\bigl(S(h)\bigr)\bigr)=f(h).\end{aligned}$$

Hence $\phi\leftharpoonup H=H^*$ and (H,ϕ) is a Frobenius algebra. Also, by assumption we have $t\leftharpoonup(h\rightharpoonup\phi)=S(h)$. It follows that $t\leftharpoonup H^*=S(H)=H$. Hence (H,t) is a Frobenius coalgebra, and so (H,ϕ,t,S) becomes a bF algebra. $\square$

Example 1.3. Let H be a finite dimensional Hopf algebra. Choose $\phi\in I_r(H^*)$ and $t\in I_r(H)$ with $\phi(t)=1$. Let σ denote the *Hopf antipode* of H, i.e., σ is the convolution inverse of id_H. Then

$$\begin{aligned}\sum\phi(t_1h)t_2 &= \sum\phi(t_1h_1)t_2h_2\sigma(h_3)\\ &= \sum\phi\bigl((th_1)_1\bigr)(th_1)_2\sigma(h_2)\quad (\Delta\text{ is multiplicative})\\ &= \sum\phi(th_1)\sigma(h_2)\quad (\phi\text{ is a right integral in }H^*)\\ &= \sum\phi(t)\varepsilon(h_1)\sigma(h_2)\quad (t\text{ is a right integral in }H)\\ &= \sigma(h)\quad (\phi(t)=1).\end{aligned}$$

Since σ is an anti-algebra and anti-coalgebra automorphism, it follows from Lemma 1.2 that (H,ϕ,t,σ) is a bF algebra.

Example 1.4. Let $B_8=k[X]/(X^8)$ (as an algebra) and $x=X \bmod (X^8)$. This is a coalgebra by defining $1\in G(B_8)$ (group-likes), $x,x^2,x^4\in P(B_8)$ (primitives) and

$$\Delta(x^3) = 1 \otimes x^3 + x \otimes x^2 + x^2 \otimes x + x^3 \otimes 1,$$
$$\Delta(x^5) = 1 \otimes x^5 + x \otimes x^4 + x^4 \otimes x + x^5 \otimes 1,$$
$$\Delta(x^6) = 1 \otimes x^6 + x^2 \otimes x^4 + x^4 \otimes x^2 + x^6 \otimes 1,$$
$$\Delta(x^7) = 1 \otimes x^7 + x \otimes x^6 + x^2 \otimes x^5 + x^3 \otimes x^4 + x^4 \otimes x^3 + x^5 \otimes x^2 + x^6 \otimes x + x^7 \otimes 1,$$
$$\varepsilon(1) = 1, \qquad \varepsilon(x^i) = 0 \quad (i = 1, \ldots, 7).$$

Define $\phi_7 \in H^*$ by $\phi_7(x^i) = \delta_{i,7}$ and $t := x^7$. Then $\sum \phi_7(t_1 x^i) t_2 = x^i$. Therefore $(B_8, \phi_7, x^7, \mathrm{id})$ is a bF algebra, again by Lemma 1.2. Clearly id is not a convolution inverse of identity. However, B_8 has the following Hopf antipode σ:

$$\sigma(1) = 1, \qquad \sigma(x) = -x, \qquad \sigma(x^2) = -x^2, \qquad \sigma(x^3) = x^3,$$
$$\sigma(x^4) = -x^4, \qquad \sigma(x^5) = x^5, \qquad \sigma(x^6) = x^6, \qquad \sigma(x^7) = -x^7.$$

Example 1.5 (Koppinen's double Frobenius algebras). Koppinen [Ko1] introduced the notion of double Frobenius algebras in 1996. It is a finite dimensional vector space which has two Frobenius algebra structures and satisfy 9 axioms. It includes the Bose–Mesner algebra attached to an association scheme. We omit the correct definition. But, roughly speaking, our bF algebras are almost equivalent to Koppinen's double Frobenius algebras under some correspondence [DT, Section 4].

Remark 1.6. Applying $\overline{S}$ (the composition inverse of S) to the formula (1) we have

$$h = \sum \overline{S}(t_2)\phi(t_1 h), \quad h \in H. \tag{3}$$

This means that $\{\overline{S}(t_2), t_1\}$ is a dual basis for ϕ. It follows by a standard argument from ring theory that

$$\sum h\overline{S}(t_2) \otimes t_1 = \sum \overline{S}(t_2) \otimes t_1 h$$

for all $h \in H$. Applying $S \otimes \mathrm{id}$ and twisting, we have

$$\sum t_1 \otimes t_2 S(h) = \sum t_1 h \otimes t_2 \tag{4}$$

for all $h \in H$. Dually we have

$$\sum \phi(x y_1) S(y_2) = \sum \phi(x_1 y) x_2 \tag{5}$$

for all $x, y \in H$. "Dually" means that (5) follows from (4) applied to the dual H^*. Note that if (H, ϕ, t, S) is any bF algebra, then (H^*, t, ϕ, S^*) is also a bF algebra.

Definition 1.7. Applying $\mathrm{id} \otimes \varepsilon$ to the formula (4) we get $t\varepsilon(h) = th$. Thus t is a right integral in H. Also for the formula (5) $y = 1$ gives $\phi(x)1 = \sum \phi(x_1)x_2$. So ϕ is a right integral in H^*. Moreover, since $1 = S(1) = \sum \phi(t_1)t_2$, we have

$\phi(t) = 1$. Since the space of right integrals is one-dimensional, an element $\phi \otimes t$ in $H^* \otimes H$ is uniquely determined from H. We call it the *integral double* for H.

Regard $H^* \otimes H$ as $\mathrm{End}(H)$ via the identification

$$(f \otimes h)(x) = f(x)h$$

for all $f \in H^*$ and $h, x \in H$. Observe that $\mathrm{Tr}(f \otimes h) = f(h)$ for $f \otimes h \in \mathrm{End}(H)$. For example, as $\mathrm{id}_H = \sum \phi \leftharpoonup t_1 \otimes \overline{S}(t_2)$ by (3), we obtain

$$\dim(H) \cdot 1_k = \phi\Big(\sum t_1 \overline{S}(t_2)\Big). \tag{6}$$

It will be shown in Theorem 2.5 that a description of the integral double is given in terms of the trace function on $\mathrm{End}(H)$.

Theorem 1.8. *Let H be a finite dimensional algebra and coalgebra satisfying* (BF1) *and* (BF2). *Let $S : H \to H$ be an anti-algebra and anti-coalgebra automorphism. Then the following are equivalent*:

(a) *There is $\phi \in H^*$ such that (H, ϕ) is a Frobenius algebra satisfying* (5).
(b) *There is $t \in H$ such that (H, t) is a Frobenius coalgebra satisfying* (4).
(c) *(H, ϕ, t, S) becomes a bF algebra for some $\phi \in H^*$ and $t \in H$.*

Proof. (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) follow from Remark 1.6.

(a) $\Rightarrow$ (c). Since $\phi \leftharpoonup H = H^*$, there is an element $t \in H$ such that $\phi \leftharpoonup t = \varepsilon$. Then

$$\begin{aligned}\sum \phi(t_1 h)t_2 &= \sum \phi(t h_1) S(h_2) \quad \text{(by (5))}\\ &= \sum \varepsilon(h_1) S(h_2) \quad (\text{by } \phi \leftharpoonup t = \varepsilon)\\ &= S(h).\end{aligned}$$

Hence, by Lemma 1.2, (H, ϕ, t, S) is a bF algebra.

(b) $\Rightarrow$ (c). Since $t \leftharpoonup H^* = H$, there is an element $\phi \in H^*$ such that $t \leftharpoonup \phi = 1$. Then

$$\begin{aligned}\sum \phi(t_1 h)t_2 &= \sum \phi(t_1) t_2 S(h) \quad \text{(by (4))}\\ &= S(h) \quad (\text{by } t \leftharpoonup \phi = 1).\end{aligned}$$

Hence, again by Lemma 1.2, (H, ϕ, t, S) is a bF algebra. □

2. BF subalgebras

Let (H, ϕ, t, S) be a bF algebra. By a *bF subalgebra* of H we understand a subalgebra and subcoalgebra K with $S(K) \subset K$ and $(K, \phi', u, S|_K)$ becomes a bF algebra for some $\phi' \in K^*$ and $u \in K$.

Theorem 2.1. *Let (H, ϕ, t, S) be a bF algebra and K a subalgebra and coalgebra with $S(K) \subset K$. Suppose that K has an Hopf antipode $\sigma : K \to K$, i.e., $\sum \sigma(y_1)y_2 = \varepsilon(y)1 = \sum y_1\sigma(y_2)$ $(y \in K)$. Then K is a bF subalgebra of H if and only if*

$$\sum \bigl(\phi \leftharpoonup t_1\sigma\bigl(\pi(t_2)_1\bigr)\bigr)\big|_K \otimes \pi(t_2)_2 = \phi' \otimes u \tag{7}$$

for some $\phi' \in K^*$ and $u \in K$, where $\pi : H \to K$ is any linear projection.

Proof. Let $(K, \phi', u, S|_K)$ be a bF algebra for some $\phi' \in K^*$ and $u \in K$. Then from (1) we have

$$\sum \phi \leftharpoonup t_1 \otimes t_2 = S \in H^* \otimes H \quad \text{and}$$
$$\sum \phi' \leftharpoonup u_1 \otimes u_2 = S|_K \in K^* \otimes K.$$

Since $S(K) \subset K$, we have in $K^* \otimes K$,

$$\sum (\phi \leftharpoonup t_1)|_K \otimes \pi(t_2) = S|_K = \sum \phi' \leftharpoonup u_1 \otimes u_2.$$

Hence

$$\sum \bigl(\phi \leftharpoonup t_1\sigma\bigl(\pi(t_2)_1\bigr)\bigr)\big|_K \otimes \pi(t_2)_2 = \sum \phi' \leftharpoonup u_1\sigma(u_2) \otimes u_3 = \phi' \otimes u.$$

Conversely, assume that $\phi' \in K^*$ and $u \in K$ satisfy (7). Then

$$\begin{aligned}
\sum \phi'(u_1 y)u_2 &= \sum \phi\bigl(t_1\sigma\bigl(\pi(t_2)_1\bigr)\pi(t_2)_2 y\bigr)\pi(t_2)_3 \\
&= \sum \phi\bigl(t_1\varepsilon\bigl(\pi(t_2)_1\bigr)y\bigr)\pi(t_2)_2 = \sum \phi(t_1 y)\pi(t_2) \\
&= \pi\Bigl(\sum \phi(t_1 y)t_2\Bigr) = \pi\bigl(S(y)\bigr) \\
&= S(y) \quad (\text{by } S(y) \in K).
\end{aligned}$$

Since $S|_K : K \to K$ is a bijection, it follows from Lemma 1.2 that K is a bF subalgebra of H. $\square$

We give in Theorem 2.5 below an interpretation of the element $\sum (\phi \leftharpoonup t_1\sigma(\pi(t_2)_1))|_K \otimes \pi(t_2)_2$ as an endomorphism of K.

Example 2.2. Let $K \subset B_8$ be the subspace generated by 1, x^2, x^4, x^6. It is a subalgebra and subcoalgebra of B_8. Let $\pi : B_8 \to K$ be the linear projection with $\pi(x) = \pi(x^3) = \pi(x^5) = \pi(x^7) = 0$. For $t = x^7$ we have

$$\sum t_1 \otimes \pi(t_2) = x^7 \otimes 1 + x^5 \otimes x^2 + x^3 \otimes x^4 + x \otimes x^6.$$

Hence

$$\begin{aligned}\sum t_1 \otimes \pi(t_2)_1 \otimes \pi(t_2)_2 = {} & x^7 \otimes 1 \otimes 1 + x^5 \otimes 1 \otimes x^2 + x^5 \otimes x^2 \otimes 1 \\ & + x^3 \otimes 1 \otimes x^4 + x^3 \otimes x^4 \otimes 1 + x \otimes 1 \otimes x^6 \\ & + x \otimes x^2 \otimes x^4 + x \otimes x^4 \otimes x^2 + x \otimes x^6 \otimes 1.\end{aligned}$$

Hence

$$\begin{aligned}\sum t_1 \sigma\big(\pi(t_2)_1\big) \otimes \pi(t_2)_2 = {} & x^7 \otimes 1 + x^5 \otimes x^2 - x^7 \otimes 1 + x^3 \otimes x^4 \\ & - x^7 \otimes 1 + x \otimes x^6 - x^3 \otimes x^4 \\ & - x^5 \otimes x^2 + x^7 \otimes 1 \\ = {} & x \otimes x^6.\end{aligned}$$

Thus $(K, (\phi_7 \leftharpoonup x)|_K, x^6, \mathrm{id})$ is a bF algebra, by Theorem 2.1.

Example 2.3. Let $B_8' = k[X]/(X^8)$ (as an algebra) and $x = X \bmod (X^8)$. Define the coalgebra structure by

$$\begin{aligned}&\Delta'(1) = 1 \otimes 1, \qquad \Delta'\big(x^i\big) = 1 \otimes x^i + x^i \otimes 1 \quad (1 \leqslant i \leqslant 6),\\ &\Delta'\big(x^7\big) = 1 \otimes x^7 + x \otimes x^6 + x^2 \otimes x^5 + x^3 \otimes x^4 + x^4 \otimes x^3 + x^5 \otimes x^2 \\ &\qquad + x^6 \otimes x + x^7 \otimes 1,\\ &\varepsilon'(1) = 1, \qquad \varepsilon'\big(x^i\big) = 0 \quad (1 \leqslant i \leqslant 7).\end{aligned}$$

Then $(B_8', \phi_7, x^7, \mathrm{id})$ is a bF algebra, where $\phi_7(x^i) = \delta_{i,7}$. The Hopf antipode $\sigma' : B_8' \to B_8'$ is given by

$$\sigma'(1) = 1, \qquad \sigma'\big(x^i\big) = -x^i \quad (1 \leqslant i \leqslant 6), \qquad \sigma'\big(x^7\big) = 5x^7.$$

Let $K' \subset B_8'$ be the subspace generated by 1, x^2, x^4, x^6, which is a subalgebra and subcoalgebra of B_8'. K' is a Frobenius algebra, but clearly it is not a Frobenius coalgebra. Indeed, any element in K' is written as $c1 + y$, where $c \in k$ and y primitive. Let $f \in H^*$. Then

$$(c1 + y) \leftharpoonup f = f(c1)1 + f(1)y + f(y)1 = f(c1 + y)1 + f(1)y.$$

Hence we have $\dim(c1 + y) \leftharpoonup H \leqslant 2$.

Thus K' does not become a bF subalgebra of B_8'. For $t = x^7$ we have

$$\sum t_1 \sigma'\big(\pi(t_2)_1\big) \otimes \pi(t_2)_2 = -2x^7 \otimes 1 + x^5 \otimes x^2 + x^3 \otimes x^4 + x \otimes x^6.$$

It is easy to see that

$$\begin{aligned}&\big(\phi_7 \leftharpoonup \big(-2x^7\big)\big)\big|_{K'} \otimes 1 + \big(\phi_7 \leftharpoonup x^5\big)\big|_{K'} \otimes x^2 + \big(\phi_7 \leftharpoonup x^3\big)\big|_{K'} \otimes x^4 \\ &\quad + \big(\phi_7 \leftharpoonup x\big)\big|_{K'} \otimes x^6\end{aligned}$$

is actually not of the form $\phi' \otimes u$.

In 1939 Nakayama [Na] gave a condition for a quotient A/I of a Frobenius algebra (A,ϕ) to be Frobenius. The next result is a variant of this.

Theorem 2.4. *Let (H,ϕ,t,S) be a bF algebra and I an ideal of H satisfying $\Delta(I)\subset I\otimes H+H\otimes I$, $\varepsilon(I)=0$, and $S(I)\subset I$. Let $\pi: H\to H/I$ be the canonical projection and $S^\pi: H/I\to H/I$ be the induced anti-algebra and anti-coalgebra automorphism, i.e., $S^\pi(\pi(h))=\pi(S(h))$. Then $(H/I,\phi',u,S^\pi)$ is a bF algebra for some $\phi'\in(H/I)^*$ and $u\in H/I$ if and only if there is an element $t'\in H$ such that $r(I)=t'H$ and*

$$\sum (ht')_1\otimes\pi\big((ht')_2\big)=\sum h_1t'\otimes\pi(h_2) \tag{8}$$

for all $h\in H$, where $r(I)$ denotes the right annihilator of I.

Proof. ($\Rightarrow$) Since $H\rightharpoonup\phi=H^*$, there exists $t'\in H$ such that $\phi'\circ\pi=t'\rightharpoonup\phi$. We show that this t' is a required one. Since

$$\phi(HIt')=\phi(It')=(t'\rightharpoonup\phi)(I)=\phi'\big(\pi(I)\big)=0,$$

we have $It'=0$ by non-degeneracy of ϕ. Hence $t'\in r(I)$ and so $t'H\subset r(I)$.

Now let $x\in l(t'H)$. Then $xt'=0$ and thus

$$\phi'\big(\pi(H)\pi(x)\big)=\phi'\big(\pi(Hx)\big)=\phi(Hxt')=0.$$

Thus $\pi(x)=0$ since $(H/I,\phi')$ is a Frobenius algebra, and therefore $l(t'H)\subset I$. We conclude (cf. [CR, (61.2) Theorem]) that

$$t'H=r\big(l(t'H)\big)\supset r(I)\quad\text{and hence}\quad r(I)=t'H.$$

Next, from (5) for H/I we have

$$\sum\phi'\big(\pi(x)\pi(y)_1\big)S^\pi\big(\pi(y)_2\big)=\sum\phi'\big(\pi(x)_1\pi(y)\big)\pi(x)_2$$

for all $x,y\in H$. It follows that

$$\begin{aligned}\sum\phi(xy_1t')S^\pi\big(\pi(y_2)\big)&=\sum\phi(x_1yt')\pi(x_2)\\&\overset{(5)}{=}\sum\phi\big(x(yt')_1\big)\pi\big(S\big((yt')_2\big)\big).\end{aligned}$$

Since $\phi\leftharpoonup H=H^*$ and S^π is a bijection, it follows

$$\sum y_1t'\otimes\pi(y_2)=(yt')_1\otimes\pi\big((yt')_2\big).$$

This proves (8).

($\Leftarrow$) Define the map ϕ' by

$$\phi': H/I\to k,\quad h+I\mapsto\phi(ht').$$

Since $t'\in r(I)$, ϕ' is well-defined. Let $x\in H$ be such that $\phi'(\pi(H)\pi(x))=0$. Since

$$\phi'\big(\pi(H)\pi(x)\big)=\phi'\big(\pi(Hx)\big)=\phi(Hxt'),$$

we have $xt' = 0$ and hence $x \in l(t'H) = l(r(I)) = I$. Thus $(H/I, \phi')$ is a Frobenius algebra.

Now we have for all $x, y \in H$,

$$\begin{aligned}\sum \phi'\bigl(\pi(x)_1\pi(y)\bigr)\pi(x)_2 &= \phi'\bigl(\pi(x_1 y)\bigr)\pi(x_2) = \sum \phi(x_1 y t')\pi(x_2)\\ &= \sum \phi\bigl(x(yt')_1\bigr)\pi\Bigl(S\bigl((yt')_2\bigr)\Bigr) \quad \text{(by (5))}\\ &= \sum \phi(xy_1 t')S^{\pi}\bigl(\pi(y_2)\bigr) \quad \text{(by (8))}\\ &= \phi'\bigl(\pi(x)\pi(y)_1\bigr)S^{\pi}\bigl(\pi(y)_2\bigr).\end{aligned}$$

Applying Theorem 1.8(b), we conclude that $(H/I, \phi', u, S^{\pi})$ is a bF algebra for some $u \in H/I$. $\square$

For the case of finite dimensional Hopf algebras, Kauffman and Radford [KR] describe the integral double $\phi \otimes t$ in terms of the trace function on $\mathrm{End}(H)$, where ϕ is a right integral in H^* and t is a right integral in H satisfying $\phi(t) = 1$. We want to prove that this is true for bF algebras with Hopf antipodes.

We let $\ell(h)$, $r(h)$ be the endomorphisms of H defined by

$$\ell(h)(x) = hx \quad \text{and} \quad r(h)(x) = xh$$

for all $h, x \in H$, and we let $\ell(f)$, $r(f)$ be the endomorphisms of H defined by

$$\ell(f)(h) = f \rightharpoonup h \quad \text{and} \quad r(f)(h) = h \leftharpoonup f$$

for all $f \in H^*$ and $h \in H$.

Theorem 2.5. *Let (H, ϕ, t, S) be a bF algebra with Hopf antipode σ. Let $P := \phi \otimes t$. Then*

(i) $f(P(h)) = \mathrm{Tr}(r(h) \circ \sigma \circ \ell(f) \circ S)$ *for all $h \in H$, $f \in H^*$. In particular,*

$$\mathrm{Tr}(\sigma \circ S) = \phi(1)\varepsilon(t).$$

(ii) *Let K be a subalgebra and subcoalgebra with $S(K) \subset K$. Let P_K be the endomorphism of* $\mathrm{End}(K)$ *defined by*

$$f\bigl(P_K(y)\bigr) = \mathrm{Tr}\bigl(r(y) \circ \sigma|_K \circ \ell(f) \circ S|_K\bigr), \quad y \in K,\ f \in K^*.$$

Then we have

$$P_K = \sum \bigl(\phi \leftharpoonup t_1\sigma\bigl(\pi(t_2)_1\bigr)\bigr)\big|_K \otimes \pi(t_2)_2 \quad \textit{and} \quad \mathrm{Tr}(P_K) = 1$$

where $\pi : H \to K$ is a linear projection.

Proof. (i) We have for $x \in H$

$$\begin{aligned}
&\big(r(h)\circ\sigma\circ\ell(f)\circ S\big)(x)\\
&\quad=\big(r(h)\circ\sigma\big)\Big(\sum\phi(t_1x)t_2f(t_3)\Big)\quad\Big(\text{by } S(x)=\sum\phi(t_1x)t_2\Big)\\
&\quad=\sum\phi(t_1x)\sigma(t_2)hf(t_3).
\end{aligned}$$

Hence

$$\mathrm{Tr}\big(r(h)\circ\sigma\circ\ell(f)\circ S\big)=\sum\phi\big(t_1\sigma(t_2)h\big)f(t_3)=\phi(h)f(t)=f\big(P(h)\big).$$

(ii) Note that $\sigma(K)\subset K$, see the proof of [Ni, Theorem 1]. Let $x\in K$. Since $S(x)\in K$, $S(x)=\pi(S(x))=\pi(\sum\phi(t_1x)t_2)=\sum\phi(t_1x)\pi(t_2)$. Thus

$$\big(r(y)\circ\sigma|_K\circ\ell(f)\circ S|_K\big)(x)=\sum\phi(t_1x)\sigma\big(\pi(t_2)_1\big)f\big(\pi(t_2)_2\big)y.$$

Hence

$$\mathrm{Tr}\big(r(y)\circ\sigma|_K\circ\ell(f)\circ S|_K\big)=\sum\phi\big(t_1\sigma\big(\pi(t_2)_1\big)y\big)f\big(\pi(t_2)_2\big).$$

Hence

$$P_K=\big(\sum\phi\leftharpoondown t_1\sigma\big(\pi(t_2)_1\big)\big)\big|_K\otimes\pi(t_2)_2$$

and thus

$$\mathrm{Tr}(P_K)=\sum\phi\big(t_1\sigma\big(\pi(t_2)_1\big)\pi(t_2)_2\big)=\sum\phi\big(t_1\varepsilon\big(\pi(t_2)\big)\big).$$

We show that there exists a linear projection $\pi:H\to K$ with $\varepsilon\circ\pi=\varepsilon$. Then we have $\mathrm{Tr}(P_K)=\sum\phi(t_1\varepsilon(t_2))=\phi(t)=1$.

Let $\{v_1=1,v_2,\dots,v_l,v_{l+1},\dots,v_n\}$ be a k-basis for H such that $\{v_1=1, v_2,\dots,v_l\}$ is a k-basis for K. If we let $v'_j=v_j-\varepsilon(v_j)1$ $(j=l+1,\dots,n)$, then

$$\big\{v_1,\dots,v_l,v'_{l+1},\dots,v'_n\big\}$$

is also a basis for H. Let $\pi:H\to K$ be the projection with $\pi(v'_{l+1})=\cdots=\pi(v'_n)=0$. Then $\varepsilon\circ\pi=\varepsilon$. □

3. Morphisms of bF algebras

Let (H,ϕ_H,t_H,S_H) and (L,ϕ_L,t_L,S_L) be bF algebras. A linear map $F: H\to L$ is called a *morphism* if F is an algebra and coalgebra map with $F\circ S_H=S_L\circ F$. Then $F^*:L^*\to H^*$ becomes a morphism. A morphism $F:H\to L$ is called *regular* if

$$\lambda:=\phi_L\big(F(t_H)\big)\neq 0.$$

Note that if $F:H\to L$ is regular, then $F^*:L^*\to H^*$ is also regular. The next result is a variant of [Ko3, Theorem 9.2, Propositions 9.7 and 11.1]. But our methods are very different.

Theorem 3.1. *Let (H, ϕ_H, t_H, S_H) and (L, ϕ_L, t_L, S_L) be bF algebras and $F: H \to L$ a regular morphism. Then we have*

$$\sum \lambda^{-1}\phi_L\big(F(t_H)_1 F(h)\big)F(t_H)_2 = S_L\big(F(h)\big) \quad \forall h \in H$$

and $(F(H), \lambda^{-1}\phi_L|_{F(H)}, F(t_H), S_L|_{F(H)})$ becomes a bF algebra, thus $F(H)$ is a bF subalgebra of L. Moreover, we have

$$F \circ \mathcal{N}_H = \mathcal{N}_L \circ F, \qquad F \circ {}^c\mathcal{N}_H = {}^c\mathcal{N}_L \circ F,$$
$$\dim\big(F(H)\big) \cdot 1_k = \lambda^{-1}\phi_L\Big(F\Big(\sum t_{H1}\overline{S}(t_{H2})\Big)\Big).$$

Proof. Let $t := t_H$. Then

$$\begin{aligned}
&\sum \lambda^{-1}\phi_L\big(F(t)_1 F(h)\big)F(t_2)\\
&\quad = \sum \lambda^{-1}\phi_L\big(F(t_1 h)\big)F(t_2)\\
&\quad = \sum \lambda^{-1}\phi_L\big(F(t_1)\big)F\big(t_2 S(h)\big) \quad \text{(by (4))}\\
&\quad = \sum \lambda^{-1}\phi_L\big(F(t_1)\big)F(t_2)S\big(F(h)\big)\\
&\quad = S\big(F(h)\big)\lambda^{-1}\Big(\sum \phi_L\big(F(t)_1\big)F(t)_2\Big)\\
&\quad = S\big(F(h)\big)\lambda^{-1}\phi_L\big(F(t)\big) \quad (\phi_L \text{ is a right integral})\\
&\quad = S\big(F(h)\big)\lambda^{-1}\lambda\\
&\quad = S\big(F(h)\big).
\end{aligned}$$

Note that $S_L|_{F(H)}$ is an anti-algebra and anti-coalgebra isomorphism of $F(H)$. Hence, by Lemma 1.2, $(F(H), \lambda^{-1}\phi_L|_{F(H)}, F(t_H), S_L|_{F(H)})$ is a bF algebra. It follows immediately that

$$\mathcal{N}_L|_{F(H)} = \mathcal{N}_{F(H)} \quad \text{and} \quad \alpha_{F(H)} \circ F = \alpha_H,$$

where $\alpha_{F(H)}$ (respectively α_H) is the right modular function on $F(H)$ (respectively H). Now

$$\begin{aligned}
F\big(\mathcal{N}_H(h)\big) &= F\overline{S}^2\Big(\sum \alpha_H(h_1)h_2\Big) \quad \text{(by [DT, 3.2(b)])}\\
&= \overline{S}^2\Big(\sum \alpha_{F(H)}\big(F(h_1)F(h_2)\big)\Big)\\
&\qquad \big(\text{by } F \circ \overline{S} = \overline{S} \circ F,\ \alpha_{F(H)} \circ F = \alpha_H\big)\\
&= \overline{S}^2\big(F(h) \leftharpoonup \alpha_F(H)\big) \quad (F \text{ is comultiplicative})\\
&= \mathcal{N}_{F(H)}\big(F(h)\big) = \mathcal{N}_L\big(F(h)\big).
\end{aligned}$$

Applying this to the regular morphism $F^*: L^* \to H^*$, we have $\mathcal{N}_{H^*} \circ F^* = F^* \circ \mathcal{N}_{L^*}$. This implies $F \circ {}^c\mathcal{N}_H = {}^c\mathcal{N}_L \circ F$.

The third equation follows immediately from (6) for $F(H)$. $\square$

Definition 3.2 (The maps F^θ, F^κ, $F^{\theta'}$, $F^{\kappa'}$). For any given linear map $F: H \to L$, we define linear maps $F^\theta, F^\kappa, F^{\theta'}, F^{\kappa'}: L \to H$ by

$$F^\theta = \theta_H^{-1} \circ F^* \circ \theta_L, \qquad F^\kappa = \kappa_H \circ F^* \circ \kappa_L^{-1},$$
$$F^{\theta'} = \theta_H'^{-1} \circ F^* \circ \theta_L', \qquad F^{\kappa'} = \kappa_H' \circ F^* \circ \kappa_L'^{-1}.$$

See Definition 1.1 for definitions of θ, θ', κ, κ'. (In [Ko2], F^θ is written as $F^\sim$, F^κ as F^-, and $F^{\theta'}$ as F^+.) These maps are characterized by

$$\phi_H\big(F^\theta(\ell)h\big) = \phi_L\big(\ell F(h)\big), \qquad \sum t_{L1} \otimes F^\kappa(t_{L2}) = \sum F(t_{H1}) \otimes t_{H2},$$
$$\phi_H\big(hF^{\theta'}(\ell)\big) = \phi_L\big(F(h)\ell\big), \qquad \sum F^{\kappa'}(t_{L1}) \otimes t_{L2} = \sum t_{H1} \otimes F(t_{H2}),$$

for all $h \in H$ and $\ell \in L$. We have the following explicit forms ($t := t_H$):

$$F^\theta(\ell) = \sum \phi_L\big(\ell F\big(\overline{S}(t_2)\big)t_1\big) \quad \text{and} \quad F^{\theta'}(\ell) = \sum \overline{S}(t_2)\phi_L\big(F(t_1)\ell\big). \tag{9}$$

Indeed, since $h = \sum \phi(h\overline{S}(t_2))t_1$ we have

$$F^\theta(\ell) = \sum \phi_H\big(F^\theta(\ell)\overline{S}(t_2)\big)t_1 = \sum \phi_L\big(\ell F\big(\overline{S}(t_2)\big)\big)t_1.$$

Example 3.3. For bF antipode $S: H \to H$ we have

$$S^\theta = \kappa' \circ \theta, \qquad S^{\theta'} = \mathcal{N} \circ {}^c\mathcal{N} \circ S. \tag{10}$$

For, $S^\theta = \theta^{-1} \circ (\kappa \circ \theta')^* \circ \theta = \theta^{-1} \circ \theta \circ \kappa' \circ \theta = \kappa' \circ \theta$, and by (2),

$$\begin{aligned} S^{\theta'} &= \theta'^{-1} \circ (\kappa \circ \theta')^* \circ \theta' = \theta'^{-1} \circ \theta \circ \kappa' \circ \theta' \\ &= \big(\theta'^{-1} \circ \theta\big) \circ \big(\kappa' \circ \kappa^{-1}\big) \circ (\kappa \circ \theta') = \mathcal{N} \circ {}^c\mathcal{N} \circ S. \end{aligned}$$

Proposition 3.4. *Let $F: H \to L$ be a linear map. Then we have*

(i) $(F^\theta)^{\theta'} = F = (F^{\theta'})^\theta$.
(ii) $F^\kappa \circ S_L = S_H \circ F^{\theta'}$.
(iii) $\mathcal{N}_H \circ F^\theta = F^{\theta'} \circ \mathcal{N}_L$.
(iv) $F^\theta = F^{\theta'} \Leftrightarrow F \circ \mathcal{N}_H = \mathcal{N}_L \circ F$.
If F is a coalgebra map with $F \circ S_H = S_L \circ F$, then this is equivalent to

$$\alpha_L \circ F = \alpha_H.$$

(v) $F^\kappa = F^{\kappa'} \Leftrightarrow F \circ {}^c\mathcal{N} = {}^c\mathcal{N} \circ F$.
If F is a algebra map with $F \circ S_H = S_L \circ F$, then this is equivalent to

$$F(\boldsymbol{a}_H) = \boldsymbol{a}_L$$

where $\boldsymbol{a}$ denotes the right modular element [DT, 1.3].
(vi) $F^\theta = F^{\kappa'} \Leftrightarrow S_L \circ F = F \circ S_H$.
(vii) $F^{\theta'} = F^\kappa \Leftrightarrow (S_L)^\theta \circ F = F \circ (S_H)^\theta$.

Proof. (i) Since $\theta^* = \theta'$ we have

$$\big(F^\theta\big)^{\theta'} = \theta'^{-1} \circ \big(\theta^{-1} \circ F^* \circ \theta\big)^* \circ \theta' = \theta'^{-1} \circ \theta' \circ F \circ \theta'^{-1} \circ \theta' = F.$$

(ii) $F^\kappa \circ S_L = \kappa \circ F^* \circ \kappa^{-1} \circ \kappa \circ \theta' = \kappa \circ \theta' \circ \theta'^{-1} \circ F^* \circ \theta' = S_H \circ F^{\theta'}$.
(iii) $\mathcal{N} \circ F^\theta = (\theta'^{-1} \circ \theta) \circ (\theta^{-1} \circ F^* \circ \theta) = \theta'^{-1} \circ F^* \circ \theta'^{-1} \circ \theta = F^{\theta'} \circ \mathcal{N}$.
(iv) We have

$$\begin{aligned} F^\theta = F^{\theta'} &\Leftrightarrow \theta^{-1} \circ F^* \circ \theta = \theta'^{-1} \circ F^* \circ \theta' \\ &\Leftrightarrow \big(\theta' \circ F \circ \theta'^{-1}\big)^* = \big(\theta \circ F \circ \theta^{-1}\big)^* \\ &\Leftrightarrow \theta' \circ F \circ \theta'^{-1} = \theta \circ F \circ \theta^{-1} \\ &\Leftrightarrow F \circ \theta'^{-1} \circ \theta = \theta'^{-1} \circ \theta \circ F \\ &\Leftrightarrow F \circ \mathcal{N} = \mathcal{N} \circ F. \end{aligned}$$

Now, assume that F is a coalgebra map with $F \circ S_H = S_L \circ F$. Recall that $\mathcal{N}(h) = \overline{S}^2(h \leftharpoonup \alpha)$ and hence $\varepsilon \circ \mathcal{N} = \alpha$. Therefore, if $F \circ \mathcal{N}_H = \mathcal{N}_L \circ F$, then

$$\alpha_L \circ F = \varepsilon_L \circ \mathcal{N}_L \circ F = \varepsilon_L \circ F \circ \mathcal{N}_H = \varepsilon_H \circ \mathcal{N}_H = \alpha_H.$$

Conversely, assume that $\alpha_L \circ F = \alpha_H$. Then

$$\begin{aligned} F\big(\mathcal{N}_H(h)\big) &= F \circ \overline{S}^2\Big(\sum \alpha_H(h_1)h_2\Big) \\ &= \overline{S}^2\Big(\sum \alpha_L\big(F(h_1)F(h_2)\big)\Big) \\ &= \overline{S}^2\big(F(h) \leftharpoonup \alpha_L\big) \quad (F \text{ is comultiplicative}) \\ &= \mathcal{N}_L\big(F(h)\big). \end{aligned}$$

(v) is the dual of (iv).
(vi) We have

$$\begin{aligned} F^\theta = F^{\kappa'} &\Leftrightarrow \theta^{-1} \circ F^*\theta = \kappa' \circ F^* \circ \kappa'^{-1} \\ &\Leftrightarrow F^* \circ \theta \circ \kappa' = \theta \circ \kappa' \circ F^* \\ &\Leftrightarrow (\kappa \circ \theta' \circ F)^* = (F \circ \kappa \circ \theta')^* \\ &\Leftrightarrow \kappa \circ \theta' \circ F = F \circ \kappa \circ \theta' \\ &\Leftrightarrow S \circ F = F \circ S. \end{aligned}$$

$$\begin{aligned} \text{(vii) } F^{\theta'} = F^\kappa &\Leftrightarrow \theta'^{-1} \circ F^* \circ \theta' = \kappa \circ F^* \circ \kappa^{-1} \\ &\Leftrightarrow F^* \circ \theta' \circ \kappa = \theta' \circ \kappa \circ F^* \\ &\Leftrightarrow (F^* \circ \theta' \circ \kappa)^* = (\theta' \circ \kappa \circ F^*)^* \\ &\Leftrightarrow \kappa' \circ \theta \circ F = F \circ \kappa' \circ \theta \\ &\Leftrightarrow S^\theta \circ F = F \circ S^\theta \quad \text{(by (10))}. \qquad \Box \end{aligned}$$

The next result is essentially due to Koppinen [Ko2, Theorem 7.1].

Theorem 3.5. *Let $F : H \to L$ be a morphism of bF algebras. Then the following are equivalent*:

(a) $F \circ \mathcal{N}_H = \mathcal{N}_L \circ F$ *and* $F \circ {}^c\mathcal{N}_H = {}^c\mathcal{N}_L \circ F$.
(b) $\alpha_L \circ F = \alpha_H$ *and* $F(\boldsymbol{a}_H) = \boldsymbol{a}_L$.
(c) $F^\theta = F^{\theta'} = F^\kappa = F^{\kappa'}$.
(d) $F^\theta \circ S_L = S_H \circ F^\theta$.

In this case (*for example, when F is regular*) *we have* $F^{\theta\theta} = F$.

Proof. (a) ⇒ (b) ⇒ (c) follow from (iv)–(vi) in Proposition 3.4.

(c) ⇒ (d) follows from (ii) in Proposition 3.4.

(d) ⇒ (a). We have

$$S^{\theta'} \circ F = S^{\theta'} \circ F^{\theta\theta'} = \left(F^\theta \circ S\right)^{\theta'} = \left(S \circ F^\theta\right)^{\theta'} = F^{\theta\theta'} \circ S^{\theta'} = F \circ S^{\theta'}.$$

Since $S^{\theta'} = \mathcal{N} \circ {}^c\mathcal{N} \circ S$ by (10) and $S \circ F = F \circ S$ it follows that

$$\mathcal{N} \circ {}^c\mathcal{N} \circ F = F \circ \mathcal{N} \circ {}^c\mathcal{N}. \tag{11}$$

Now since $\boldsymbol{a}$ is group-like and $\overline{S}^2$-stable [DT, 3.3(d)],

$$(\mathcal{N} \circ {}^c\mathcal{N} \circ F)(1_H) = \overline{S}^2(\boldsymbol{a}_L \leftharpoonup \alpha_L) = \alpha_L(\boldsymbol{a}_L)\boldsymbol{a}_L,$$

and

$$(F \circ \mathcal{N} \circ {}^c\mathcal{N})(1_H) = F \circ \overline{S}^2(\boldsymbol{a}_H \leftharpoonup \alpha_H) = \alpha_H(\boldsymbol{a}_H)F(\boldsymbol{a}_H).$$

Thus we have $\alpha_L(\boldsymbol{a}_L)\boldsymbol{a}_L = \alpha_H(\boldsymbol{a}_H)F(\boldsymbol{a}_H)$, and applying ε we have $\boldsymbol{a}_L = F(\boldsymbol{a}_H)$. (In this argument we used $\varepsilon(F(\boldsymbol{a}_H)) = \varepsilon(\boldsymbol{a}_H) = 1$.) This implies ${}^c\mathcal{N} \circ F = F \circ {}^c\mathcal{N}$, by (v) in Proposition 3.4. Hence from (11) we have $\mathcal{N} \circ F \circ {}^c\mathcal{N} = F \circ \mathcal{N} \circ {}^c\mathcal{N}$ and so $\mathcal{N} \circ F = F \circ \mathcal{N}$. □

Remark 3.6. Note that F^θ does not become a morphism even if F is a morphism. However we have

$$\begin{aligned}
&\phi_L\big(\ell F(hh')\big) = \phi_H\big(F^\theta(\ell)hh'\big) \quad \big(= \phi_H\big(\mathcal{N}^{-1}(h')F^\theta(\ell)h\big)\big),\\
&\phi_L\big(\ell F(h)F(h')\big) = \phi_H\big(F^\theta\big(\ell F(h)\big)h'\big),\\
&\phi_L\big(\ell F(h)F(h')\big) = \phi_L\big(\mathcal{N}^{-1}\big(F(h')\big)\ell F(h)\big)\\
&\quad \big(= \phi_H\big(F^\theta\big(\mathcal{N}^{-1}\big(F(h')\big)\ell\big)h\big)\big).
\end{aligned}$$

These imply

$$F^\theta(\ell)h = F^\theta\big(\ell F(h)\big) \quad \text{and} \quad F^\theta\big(\mathcal{N}^{-1}\big(F(h)\big)\ell\big) = \mathcal{N}^{-1}(h)F^\theta(\ell) \tag{12}$$

for all $\ell \in L$ and $h \in H$, since the left sides are equal and the bilinear form $H \times H \to k$, $(h, h') \mapsto \phi_H(hh')$ is non-degenerate. Let $(K, \phi', u, S|_K)$ be a bF subalgebra of a bF algebra (H, ϕ, t, S). Let $\iota : K \to H$ be the inclusion. Then, by (9), $\iota^\theta : H \to K$ is given by

$$\iota^\theta(h) = \sum \phi\big(h_1 \overline{S}(u)\big) h_2 \overset{(5)}{=} \sum \phi\big(h \overline{S}(u_2)\big) u_1.$$

From (12) we obtain that ι^θ is a right K-module map and

$$\iota^\theta(yh) = \beta(y)\iota^\theta(h), \quad y \in K,\ h \in H,$$

where $\beta = (\mathcal{N}_K)^{-1} \circ \mathcal{N}_H$. This fact is used in a proof of the result (due to Schneider) that any pair $K \subset H$ of finite dimensional Hopf algebras is a β-Frobenius extension, see [FMS, Theorem 1.7].

Acknowledgment

The author thanks the referee for valuable comments to improve the paper.

References

[CR] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York, 1962.

[DT] Y. Doi, M. Takeuchi, Bi-Frobenius algebras, in: Contemp. Math., Vol. 267, 2000, pp. 67–97.

[FMS] D. Fischman, S. Montgomery, H.-J. Schneider, Frobenius extensions of subalgebras of Hopf algebras, Trans. Amer. Math. Soc. 349 (1997) 4857–4895.

[KR] L.H. Kauffman, D.E. Radford, On two proofs for the existence and uniqueness of integrals for finite-dimensional Hopf algebras, in: Contemp. Math., Vol. 267, 2000, pp. 177–194.

[Ko1] M. Koppinen, On algebras with two multiplications, including Hopf algebras and Bose–Mesner algebras, J. Algebra 182 (1996) 256–273.

[Ko2] M. Koppinen, Morphisms of double Frobenius algebras I, Comm. Algebra 28 (2000) 1845–1860.

[Ko3] M. Koppinen, Morphisms of double Frobenius algebras II, Comm. Algebra 29 (2001) 2523–2540.

[Na] T. Nakayama, On Frobenius algebras I, Ann. of Math. 40 (1939) 611–633.

[Ni] W. Nichols, Quotients of Hopf algebras, Comm. Algebra 6 (1978) 1789–1800.